The Lewis Correspondence for submodular groups

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1 Introduction

Let G denote the group $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$. Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ be the modular group. A *submodular group* is a subgroup of Γ of finite index. It is the aim of this note to extend the Lewis Correspondence [5, 6, 7] from Γ to submodular groups. Since any submodular group Λ contains a submodular subgroup which is normal in Γ we will first assume that Λ is normal and only later move from Λ an arbitrary subgroup containing Λ. Let $\mathbb{H}^+ = \{x + iy \in \mathbb{C} : y > 0\}$ be the upper half plane in $\mathbb{C}$. The group G acts on $\mathbb{H}^+$ by linear fractions, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \dfrac{az+b}{cz+d}$. This action preserves the hyperbolic geometry given by the Riemannian metric $\frac{1}{y^2}(dx^2 + dy^2)$ so it commutes with the hyperbolic Laplace operator $\Delta = -y^2\left((\frac{\partial}{\partial x})^2 + (\frac{\partial}{\partial y})^2\right)$ and preserves the hyperbolic volume form $dx\,dy/y^2$. A *Maaß form* for Λ is a function $f \in L^2(\Lambda\backslash\mathbb{H}^+)$ which is an eigenfunction of Δ. We also define $\mathbb{H}^-$ to be the lower half plane in $\mathbb{C}$. The Lewis Correspondence attaches a certain "period function" to a given Maaß form for Γ. To extend it to $\Lambda \neq \Gamma$ we have to start with Maaß forms for Λ. These form a module under the finite group Γ/Λ under the regular representation and so Maaß forms for Λ are related to Maaß forms for Γ twisted by a finite dimensional representation (η, V_η) by the following mechanism:

Let W be a $\mathbb{C}[\Gamma]$-module, which is finite dimensional as $\mathbb{C}$-vector space and trivially acted upon by Λ. Under the action of the finite group Γ/Λ the module W decomposes into isotypic components,

$$W = \bigoplus_{\eta \in \widehat{\Gamma/\Lambda}} W(\eta), \tag{1}$$

where $\widehat{\Gamma/\Lambda}$ denotes the set of isomorphism classes of irreducible unitary representations of Γ/Λ, i.e., the unitary dual of this finite group. For $\eta \in \widehat{\Gamma/\Lambda}$ let $\breve{\eta}$ denote its dual representation. There is a natural isomorphism

$$\mathrm{ev}\colon (W \otimes \eta)^\Gamma \otimes \breve{\eta} \;\to\; W(\breve{\eta}) \tag{2}$$

given by $\mathrm{ev}(\sum_j (w_j \otimes \alpha_j) \otimes \beta) := \sum_j \langle \alpha_j, \beta\rangle\, w_j$. On the other hand, the inclusion $W(\breve{\eta}) \subset W(\eta)$ induces an isomorphism $(W \otimes \eta)^\Gamma \cong (W(\breve{\eta}) \otimes \eta)^\Gamma$

and the projection map Pr from $W \otimes \eta$ to $(W \otimes \eta)^\Gamma$ is explicitly given by

$$\Pr(w \otimes \alpha) \;=\; \frac{1}{|\Gamma/\Lambda|} \sum_{\gamma:\Gamma/\Lambda} \gamma.w \otimes \gamma.\alpha.$$

Finally, elementary character theory shows that the canonical projection $\mathcal{P}_{\breve{\eta}}\colon W \to W(\breve{\eta})$ given by the decomposition (1) equals

$$\mathcal{P}_{\breve{\eta}} w = \frac{d_\eta}{|\Gamma/\Lambda|} \sum_{\gamma\in\Gamma/\Lambda} \operatorname{tr}\eta(\gamma)\ (\gamma \cdot w), \tag{3}$$

where d_η ist the degree of η and $\breve{\eta}$. Here we have used the convention that we write the space of a representation with the same symbol as the representation itself. Occasionally, to put emphasis on the space rather than the representation, we will also write V_η for the representation space of η. In order to describe W we decompose it into isotypic components and each such component is described by $(W \otimes \eta)^\Gamma$. We will in particular apply this to the space of Maaß forms for Λ with a given Laplace eigenvalue. But we also can retrieve Maaß forms of an arbitrary submodular group Σ. For this let $\Lambda \subset \Sigma \subset \Gamma$ be a submodular group which is normal in Γ, and let W be the space of Λ-Maaß forms. Then

$$W \;\cong\; \bigoplus_{\eta\in\widehat{\Lambda/\Gamma}} (W \otimes \eta)^\Gamma \otimes \breve{\eta}.$$

The space of Σ-Maaß forms is just the space of Σ-invariants herein, i.e., the space

$$W^\Sigma \;\cong\; \bigoplus_{\eta\in\widehat{\Lambda/\Gamma}} (W \otimes \eta)^\Gamma \otimes \breve{\eta}^\Sigma.$$

So W^Σ is described by the spaces $(W \otimes \eta)^\Gamma$ and the dimensions $\dim(\breve{\eta}^\Sigma$ for $\eta \in \widehat{\Lambda/\Gamma}$. This applies in particular to the congruence subgroups $\Lambda = \Gamma(N)$ and $\Sigma = \Gamma_0(N)$. So we fix an irreducible representation η of Γ with finite image.

We fix the following notation for the canonical generators of Γ:

$$S \;=\; \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad T \;=\; \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then $S^2 = \mathbf{1} = (ST)^3$, and T is of infinite order. Let $\mathcal{F}_\eta$ be the space of holomorphic functions $f\colon \mathbb{C} \smallsetminus \mathbb{R} \to V_\eta$ with

$$\begin{aligned} f(z+1) &= \eta(T)f(z), & (4)\\ f(z) &= O(1) \quad \text{as } |\mathrm{Im}(z)| \to \infty, & (5)\\ 0 &= f(i\infty) + f(-i\infty). & (6) \end{aligned}$$

The last condition needs explaining. Since η has finite image, there is a smallest $N := N_\eta \in \mathbb{N}$ such that $\eta(T^N)$ equals the identity. It follows that f has a Fourier expansion

$$f(z) = \sum_{k\in\mathbb{Z}} e^{2\pi i \frac{k}{N} z} v_k^+, \qquad v_k^+ \in V_\eta,$$

in $\mathbb{H}^+$ and similarly with $v_k^- \in V_\eta$ in $\mathbb{H}^-$. Condition (5) leads to $v_{-k}^+ = v_k^- = 0$ for every $k \in \mathbb{N}$. Thus the limits do exist and satisfy $f(i\infty) = v_0^+$ and $f(-i\infty) = v_0^-$.

Consider the space $\mathcal{F}_{\nu,\eta}$ of all $f \in \mathcal{F}_\eta$ for which the map

$$z \mapsto f(z) - z^{-2\nu-1}\,\eta(S) f\left(\frac{-1}{z}\right) \tag{7}$$

extends holomorphically to $\mathbb{C} \smallsetminus (-\infty, 0]$ and the space $\Psi_{\nu,\eta}$ of all holomorphic functions $\psi\colon \mathbb{C} \smallsetminus (-\infty, 0] \to V_\eta$ satisfying

$$\eta(T)\psi(z) = \psi(z+1) + (z+1)^{-2\nu-1}\eta(ST^{-1})\psi\left(\frac{z}{z+1}\right) \tag{8}$$

and

$$\begin{aligned} 0 \;=\; & e^{-\pi i\nu} \lim_{\mathrm{Im}(z)\to\infty} \left(\psi(z) + z^{-2\nu-1}\eta(S)\psi\left(\frac{-1}{z}\right)\right) + \\ & + e^{\pi i\nu} \lim_{\mathrm{Im}(z)\to-\infty} \left(\psi(z) + z^{-2\nu-1}\eta(S)\psi\left(\frac{-1}{z}\right)\right), \end{aligned} \tag{9}$$

where both limits exist. We call (8) the *Lewis equation.*

Let π_ν be the principal series representation of G associated with the parameter $\nu \in \mathbb{C}$ and $\pi_\nu^{-\omega}$ the corresponding space of hyperfunction vectors. As a crucial tool we will use the space

$$A_{\nu,\eta}^{-\omega} = (\pi_\nu^{-\omega} \otimes \eta)^\Gamma = H^0(\Gamma, \pi_\nu^{-\omega} \otimes \eta) \tag{10}$$

and call it the space of η*-automorphic hyperfunctions*.

Generalizing results of Bruggeman (see [1], Prop. 2.1 and Prop. 2.3), we will show in Proposition 2.2 that there is a linear isomorphism $A^{-\omega}_{\nu,\eta} \to \mathcal{F}_{\nu,\eta}$ and (using this) establish in Proposition 2.3 a linear map

$$B\colon A^{-\omega}_{\nu,\eta} \to \Psi_{\nu,\eta},$$

which we call the *Bruggeman transform*. It turns out to be bijective unless $\nu \in \frac{1}{2} + \mathbb{Z}$.

Recall that a *Maaß wave form* for a subgroup Λ of Γ (not necessarily normal) and parameter $\nu \in \mathbb{C}$ is a function u on $\mathbb{H}^+$ which is twice continuously differentiable and satisfies

$$\begin{aligned} u(\gamma z) &= u(z) \quad \text{for every } \gamma \in \Gamma, &(11)\\ \infty &> \int_{\Gamma\backslash\mathbb{H}^+} |u(z)|^2\, dz, &(12)\\ \Delta u &= \left(\tfrac{1}{4} - \nu^2\right) u. &(13)\end{aligned}$$

By the regularity of solutions of elliptic differential equations the last condition implies that u is real analytic. Let $\mathcal{M}_\nu = \mathcal{M}_\nu^\Lambda$ be the space of all Maaß wave forms for Λ.

If Λ is normal of finite index in Γ the finite group Γ/Λ acts on this space, and as in (1) we get an isotypic decomposition,

$$\mathcal{M}_\nu \;=\; \bigoplus_{\eta \in \widehat{\Gamma/\Lambda}} \mathcal{M}_\nu(\eta). \tag{14}$$

and for each η,

$$\mathcal{M}_\nu(\breve{\eta}) \;\cong\; V_\eta^* \otimes \left(V_\eta \otimes \mathcal{M}_\nu\right)^{\Gamma/\Lambda}.$$

We set $\mathcal{M}_{\nu,\eta}$ equal to $\left(V_\eta \otimes \mathcal{M}_\nu\right)^{\Gamma/\Lambda}$. Then $\mathcal{M}_{\nu,\eta}$ can be viewed as the space of all functions $u\colon \mathbb{H}^+ \to V_\eta$ which are twice continuously differentiable and satisfy

$$\begin{aligned} u(\gamma z) &= \eta(\gamma) u(z) \quad \text{for every } \gamma \in \Gamma, &(15)\\ \infty &> \int_{\Gamma\backslash\mathbb{H}^+} \|u(z)\|^2\, dz, &(16)\\ \Delta u &= \left(\tfrac{1}{4} - \nu^2\right) u. &(17)\end{aligned}$$

We define the space $\mathcal{S}_{\nu,\eta}$ of *Maaß cusp forms* to be the space of all $u \in \mathcal{M}_{\nu,\eta}$ such that

$$\int_0^N u(z+t)\,dt \;=\; 0 \tag{18}$$

for every $z \in \mathbb{H}^+$. Here, as before, N is the order of $\eta(T)$, so that in particular $\eta(T)^N = \mathbf{1}$ and $u(z+N) = u(z)$.

For $\mathrm{Re}\nu > -\frac{1}{2}$ consider the space $\Psi^o_{\nu,\eta}$ of all $\psi \in \Psi_{\nu,\eta}$ satisfying

$$\psi(z) \;=\; O(\min\{1, |z|^{-C}\}) \qquad \text{for } z \in \mathbb{C} \setminus (-\infty, 0], \tag{19}$$

for some $0 < C < 2\mathrm{Re}\nu + 1$. We call the elements of $\Psi^o_{\nu,\eta}$ *period functions.* In Lemma 3.1 we establish for $\mathrm{Re}\nu > -\frac{1}{2}$ two linear maps $\mathcal{S}_{\nu,\eta} \to \mathcal{F}_{\nu,\eta}$ and $\mathcal{S}_{\nu,\eta} \to \Psi^o_{\nu,\eta}$.

Let $\tilde{\mathcal{M}}_{\nu,\eta}$ be the space of all functions u satisfying only (15) and (17). So there is no growth restriction on elements of $\tilde{\mathcal{M}}_{\nu,\eta}$. For an automorphic hyperfunction $\alpha \in A^{-\omega}_{\nu,\eta}$ we consider the function $u\colon G \to V_\eta$ given by

$$u(g) \;:=\; \langle \pi_{-\nu}(g)\varphi_0, \alpha\rangle\,.$$

Then u is right K-invariant, hence can be viewed as a function on $\mathbb{H}^+$. As such it lies in $\tilde{\mathcal{M}}_{\nu,\eta}$ since α is Γ-equivariant and the Casimir operator on G, which induces Δ, is scalar on π_ν with eigenvalue $\frac{1}{4} - \nu^2$. The transform $P\colon \alpha \mapsto u$ is called the *Poisson transform.* It follows from [8], Theorem 5.4.3, that the Poisson transform

$$P\colon A^{-\omega}_{\nu,\eta} \;\to\; \tilde{\mathcal{M}}_{\nu,\eta} \tag{20}$$

is an isomorphism for $\nu \notin \frac{1}{2} + \mathbb{Z}$.

For $\nu \notin \frac{1}{2}+\mathbb{Z}$ we finally define the *Lewis transform* as the map $L\colon \mathcal{M}_{\nu,\eta} \to \Psi_{\nu,\eta}$, given by

$$L \;:=\; B \circ P^{-1}. \tag{21}$$

Our first main result (see Theorem 3.3) is a generalization of [7], Thm. 1.1, and says that the Lewis transform for $\nu \notin \frac{1}{2} + \mathbb{Z}$ and $\mathrm{Re}\nu > -\frac{1}{2}$ is a linear isomorphism between the space of Maaß cusp forms $\mathcal{S}_{\nu,\eta}$ and the space $\Psi^o_{\nu,\eta}$ of period functions.

A holomorphic function on $\mathbb{C} \setminus (-\infty, 0]$ is uniquely determined by its values in $\mathbb{R}^+ := (0, \infty)$. Thus, in principle, it is possible to describe the period functions as a space of real analytic functions on the positive halfline.

Following ideas from [7], Chap. III, in this section we show how this can be done in an explicit way.

Consider the space $\Psi^{\mathbb{R}}_{\nu,\eta}$ of all real analytic functions ψ from $(0,\infty)$ to V_η satisfying

$$\begin{aligned} \eta(T)\psi(x) &= \psi(x+1) + (x+1)^{-2\nu-1}\eta(ST^{-1})\psi\left(\frac{x}{x+1}\right) && (22) \\ \psi(x) &= o(1/x), \quad \text{as } x \to 0, x > 0, && (23) \\ \psi(x) &= o(1), \quad \text{as } x \to +\infty, x \in \mathbb{R}. && (24) \end{aligned}$$

Our second main result (see Theorem 4.4) is a generalization of [7], Thm. 2, and says that for $\mathrm{Re}\nu > -\frac{1}{2}$ we have $\Psi^{\mathbb{R}}_{\nu,\eta} = \{\psi|_{(0,\infty)} : \psi \in \Psi^{o}_{\nu,\eta}\}$.

We summarize the various spaces and mappings considered so far in one diagram:

$$\begin{tikzcd}
\mathcal{M}_{\nu,\eta} \arrow[d] & \mathcal{S}_{\nu,\eta} \arrow[l] \arrow[r, "L"] \arrow[d] & \Psi^{o}_{\nu,\eta} \arrow[r, "\mathrm{res}"] \arrow[d] & \Psi^{\mathbb{R}}_{\nu,\eta} \\
\tilde{\mathcal{M}}_{\nu,\eta} & \mathcal{F}_{\nu,\eta} \arrow[r] & \Psi_{\nu,\eta} & \\
 & A^{-\omega}_{\nu,\eta} \arrow[ul, "P"] \arrow[u, "\cong"'] \arrow[ur, "B"'] & &
\end{tikzcd}$$

2 Automorphic hyperfunctions

Let A denote the subgroup of G consisting of diagonal matrices and let N be the subgroup of upper triangular matrices with ± 1 on the diagonal. Let $P = AN$ be the group of upper triangular matrices. Finally, let $K = \mathrm{PSO}(2) = \mathrm{SO}(2)/\{\pm 1\}$ be the canonical maximal compact subgroup of G. The group G then as a manifold is a direct product $G = ANK = PK$. For $\nu \in \mathbb{C}$ and $a = \pm\mathrm{diag}(t, t^{-1})$, $t > 0$, let $a^\nu = t^{2\nu}$. We insert the factor 2 for compatibility reasons. Let (π_ν, V_{π_ν}) denote the principal series representation of G with parameter ν. The representation space V_{π_ν} is the Hilbert space of all

functions $\varphi\colon G \to \mathbb{C}$ with $\varphi(anx) = a^{\nu+\frac{1}{2}}\varphi(x)$ for $a \in A, n \in N, x \in G$, and $\int_K |\varphi(k)|^2\, dk < \infty$ modulo nullfunctions. The representation is $\pi_\nu(x)\varphi(y) = \varphi(yx)$. There is a special vector φ_0 in V_{π_ν} given by

$$\varphi_0(ank) \;=\; a^{\nu+\frac{1}{2}}.$$

This vector is called the *basic spherical function* with parameter ν. The group G acts on the complex projective line $\mathbb{P}_1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ by linear fractions. This action has three orbits: the upper half plane $\mathbb{H}^+$, the lower half plane $\mathbb{H}^-$ and the real projective line $\mathbb{P}_1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$. The upper half plane can be identified with G/K via $gK \mapsto g.i$ and $\mathbb{P}_1(\mathbb{R})$ can be identified with $P\backslash G$ via

$$P\begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\mapsto\; [c:d].$$

Our embedding of $\mathbb{R}$ into $\mathbb{P}_1(\mathbb{R})$ is via $x \mapsto [1:x]$, which can be viewed as the map

$$\begin{array}{ccc} N & \to & P\backslash PwN \\ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} & \mapsto & P\begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} \end{array}$$

with the Weyl group element $w = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Note that V_{π_ν} can also be viewed as a space of sections of a line bundle over $P\backslash G$. For this bundle the above embedding provides a trivialization over $\mathbb{R}$. Using the corresponding Bruhat decomposition

$$\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm\begin{pmatrix} c^{-1} & a \\ 0 & c \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & \frac{d}{c} \end{pmatrix}$$

for $c \neq 0$ we obtain a realization of V_{π_ν} on $L^2(\mathbb{R}, \frac{1}{\pi}(1+x^2)^{2\nu}dx)$ with the action

$$\pi_\nu\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x) = (cx-a)^{-2\nu-1} f\left(\frac{dx-b}{cx-a}\right).$$

Transferring the action to $L^2(\mathbb{R}, \frac{1}{\pi}\frac{dx}{1+x^2})$ then yields the action

$$\pi_\nu\begin{pmatrix} a & b \\ c & d \end{pmatrix}\varphi(x) \;=\; \left(\frac{1+x^2}{(cx-a)^2+(dx-b)^2}\right)^{\nu+\frac{1}{2}} \varphi\left(\frac{dx-b}{-cx+a}\right)$$

used in [1]. This is the realization of the principal series we shall work with. Note that in this realization the basic spherical function is simply the constant function 1.

Let $\pi_\nu^\omega \subset \pi_\nu^{-\omega}$ be the sets of analytic vectors and hyperfunction vectors, respectively. For any open neighbourhood U of $\mathbb{P}_1(\mathbb{R})$ inside $\mathbb{P}_1(\mathbb{C})$ the space $\pi_\nu^{-\omega}$ can be identified with the space

$$\mathcal{O}(U \smallsetminus \mathbb{P}_1(\mathbb{R}))/\mathcal{O}(U),$$

where $\mathcal{O}$ denotes the sheaf of holomorphic functions. This space does not depend on the choice of U. For $U \subseteq \mathbb{C}$ this follows from Lemma 1.1.2 of [8] and generally by subtracting the Laurent series at infinity. The G-action is given by the above formula, where x is replaced by a complex variable z. Note that any hyperfunction α on $\mathbb{P}_1(\mathbb{R})$ has a restriction to $\mathbb{R}$ which can be represented by a holomorphic function on $\mathbb{C} \smallsetminus \mathbb{R}$.

Proposition 2.1 (Symmetry of gluing conditions) *For $f \in \mathcal{F}_\eta$ the following conditions are equivalent:*

(1) $z \mapsto f(z) - z^{-2\nu-1}\,\eta(S) f\left(\frac{-1}{z}\right)$ *extends holomorphically to* $\mathbb{C} \smallsetminus (-\infty, 0]$.

(2) $z \mapsto (1+z^{-2})^{\nu+\frac{1}{2}} f\left(\frac{-1}{z}\right)$ *and* $z \mapsto (1+z^2)^{\nu+\frac{1}{2}}\,\eta(S) f(z)$ *define the same hyperfunction on* $\mathbb{R} \smallsetminus \{0\}$.

Proof: "(2)⇒(1)" Suppose that

$$(1+z^{-2})^{\nu+\frac{1}{2}} f\left(\frac{-1}{z}\right) = (1+z^2)^{-\nu-\frac{1}{2}}\,\eta(S) f(z) + q(z)$$

with a function q that is holomorphic in a neighborhood of $\mathbb{R} \smallsetminus \{0\}$. For $\mathrm{Re} z > 0$ we can divide the equation by $(1+z^2)^{\nu+\frac{1}{2}}$ and obtain

$$z^{-2\nu-1} f\left(\frac{-1}{z}\right) = \eta(S) f(z) + (1+z^2)^{-\nu-\frac{1}{2}}\, q(z).$$

Since $\eta(S) = \eta(S)^{-1}$, this implies the claim.

"(1)⇒(2)" If (1) holds, by the same calculation as above we see that for $\mathrm{Re} z > 0$ the function

$$z \mapsto (1+z^2)^{\nu+\frac{1}{2}}\,\eta(S) f(z) - (1+z^{-2})^{\nu+\frac{1}{2}} f\left(\frac{-1}{z}\right)$$

extends holomorphically to the entire right halfplane. But then the symmetry of this expression under the transformation $z \mapsto -\frac{1}{z}$ yields the holomorphic extendability also on the left halfplane which proves (2). $\square$

Recall the space $A^{-\omega}_{\nu,\eta} = (\pi^{-\omega}_\nu \otimes \eta)^\Gamma = H^0(\Gamma, \pi^{-\omega}_\nu \otimes \eta)$ of η-automorphic hyperfunctions from (10).

Proposition 2.2 (cf. [1], Prop. 2.1) *There is a bijective linear map*

$$\begin{array}{rcl} A^{-\omega}_{\nu,\eta} & \to & \mathcal{F}_{\nu,\eta} \\ \alpha & \mapsto & f_\alpha \end{array}$$

such that the function $z \mapsto (1+z^2)^{\nu+\frac{1}{2}} f_\alpha(z)$ *represents the restriction* $\alpha|_{\mathbb{R}}$.

Proof: The space $A^{-\omega}_{\nu,\eta} = (\pi^{-\omega}_\nu \otimes \eta)^\Gamma$ can be viewed as the space of all V_η-valued hyperfunctions α in $\mathbb{P}_1(\mathbb{R})$ satisfying the invariance condition

$$\pi_\nu(\gamma^{-1})\alpha = \eta(\gamma)\alpha$$

for every $\gamma \in \Gamma$. Pick a representative f for α. The V_η-valued function $F\colon z \mapsto (1+z^2)^{-\nu-\frac{1}{2}} f(z)$ is holomorphic on $0 < |\mathrm{Im}(z)| < \varepsilon$ for some $\varepsilon > 0$. Note that the invariance of α under T implies that for some function q, holomorphic on a neighbourhood of $\mathbb{R}$, we have

$$\begin{aligned} \eta(T)f(z) + q(z) &= \Big(\pi_\nu(T^{-1})f\Big)(z) \\ &= \left(\frac{1+z^2}{1+(z+1)^2}\right)^{\nu+\frac{1}{2}} f(z+1) \\ &= (1+z^2)^{\nu+\frac{1}{2}} F(z+1), \end{aligned}$$

so that

$$F(z+1) = \eta(T)F(z) + (1+z^2)^{-(\nu+\frac{1}{2})} q(z).$$

Therefore F represents a hyperfunction on $\mathbb{P}_1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ which is invariant under the translation $z \mapsto z+N$. This hyperfunction has a representative which is holomorphic in $\mathbb{P}_1(\mathbb{C}) \setminus \mathbb{P}_1(\mathbb{R})$. The freedom in this representative is an additive constant. So there is a unique representative f_α of the form

$$f_\alpha(z) = \begin{cases} \dfrac{1}{2} v_0 + \displaystyle\sum_{k=1}^\infty e^{2\pi i \frac{k}{N} z} v_k^+, & z \in \mathbb{H}^+, \\ -\dfrac{1}{2} v_0 - \displaystyle\sum_{k=1}^\infty e^{-2\pi i \frac{k}{N} z} v_k^-, & z \in \mathbb{H}^-. \end{cases}$$

So $f_\alpha \in \mathcal{F}_\eta$ and $(1+z^2)^{\nu+\frac{1}{2}} f_\alpha(z)$ represents $\alpha|_{\mathbb{R}}$. To show the injectivity of the map in the Proposition assume that $f_\alpha = 0$. Then α is supported in $\{\infty\}$. Since latter set is not Γ-invariant, α must be zero. To see that f_α lies in $\mathcal{F}_{\nu,\eta}$, recall that the invariance of α under S implies that

$$(1+z^{-2})^{\nu+\frac{1}{2}} f_\alpha\left(\frac{1}{-z}\right) \;=\; (1+z^2)^{\nu+\frac{1}{2}}\eta(S) f_\alpha\,(z) + \tilde{q}(z)$$

with $\tilde{q}(z)$ holomorphic on a neighbourhood of $\mathbb{R} \smallsetminus \{0\}$. Thus Proposition 2.1 shows that f_α satisfies (7) and hence $f_\alpha \in \mathcal{F}_{\nu,\eta}$. To finally show surjectivity, let $f \in \mathcal{F}_{\nu,\eta}$. Then the function

$$z \mapsto (1+z^2)^{\nu+\frac{1}{2}} f(z)$$

represents a hyperfunction β_0 on $\mathbb{R}$ that satisfies $\pi_\nu(T^{-1})\beta_0 = \eta(T)\beta_0$. Let $\beta_\infty := (\pi_\nu \otimes \eta)(S)\beta_0$. Then β_∞ is a hyperfunction on $\mathbb{P}_1(\mathbb{R}) \smallsetminus \{0\}$ with representative $z \mapsto (1+z^{-2})^{\nu+\frac{1}{2}}\eta(S) f(\frac{-1}{z})$. According to Proposition 2.1 the restrictions of β_0 and β_∞ to $\mathbb{P}_1(\mathbb{R}) \smallsetminus \{0,\infty\}$ agree. Thus β_0 and β_∞ are restrictions of a hyperfunction β on $\mathbb{P}_1(\mathbb{R})$ which is then easily seen to be S-invariant. Using β_0 we see that the support of

$$(\pi_\nu \otimes \eta)(T)\beta - \beta$$

is contained in $\{\infty\}$. Using β_∞ we see that for $|z| > 2$, $z \notin \mathbb{R}$, this hyperfunction is represented by

$$\begin{aligned}
&\left(\frac{1+z^2}{1+(z-1)^2}\right)^{\nu+\frac{1}{2}} \left(1+(z-1)^{-2}\right)^{\nu+\frac{1}{2}} \eta(TS) f\left(\frac{-1}{z-1}\right) \\
&\qquad\qquad -(1+z^{-2})^{\nu+\frac{1}{2}}\eta(S) f\left(\frac{-1}{z}\right) = \\
&\quad = \;\; (1+z^2)^{\nu+\frac{1}{2}}(z-1)^{-2\nu-1}\eta(TS) f\left(\frac{-1}{z-1}\right) \\
&\qquad\qquad -(1+z^{-2})^{\nu+\frac{1}{2}}\eta(S) f\left(\frac{-1}{z}\right) \\
&\quad = \;\; (1+z^{-2})^{\nu+\frac{1}{2}} \times \\
&\qquad \left(\left(\frac{z}{z-1}\right)^{2\nu+1} \eta(TST^{-1}) f\left(\frac{z-2}{z-1}\right) - \eta\left(ST^{-1}\right) f\left(\frac{z-1}{z}\right)\right)
\end{aligned}$$

Since $f(z)$ is holomorphic around $z = 1$ it follows that this function is holomorphic around $z = \infty$. Hence β is invariant under T. Now the claim follows because the elements S and T generate Γ. $\square$

Proposition 2.3 (Bruggeman transform; cf. [1], Prop. 2.3) *For $\alpha \in A^{-\omega}_{\nu,\eta}$ put*

$$\psi_\alpha(z) \ := \ f_\alpha(z) - z^{-2\nu-1}\eta(S) f_\alpha\left(\frac{-1}{z}\right),$$

with f_α as in Proposition 2.2. Then the Bruggeman transform $B\colon \alpha \mapsto \psi_\alpha$ *maps $A^{-\omega}_{\nu,\eta}$ to $\Psi_{\nu,\eta}$. It is a bijection if $\nu \notin \frac{1}{2} + \mathbb{Z}$.*

Proof: Let $\alpha \in A^{-\omega}_{\nu,\eta}$ and define ψ_α as in the Proposition. By Proposition 2.2 the map ψ_α extends to $\mathbb{C} \setminus (-\infty, 0]$. We compute

$$\begin{aligned}
&\psi_\alpha(z+1) + (z+1)^{-2\nu-1}\eta(ST^{-1})\psi_\alpha\left(\frac{z}{z+1}\right) = \\
&= \ f_\alpha(z+1) - (z+1)^{-2\nu-1}\eta(S) f_\alpha\left(\frac{-1}{z+1}\right) + (z+1)^{-2\nu-1}\eta(ST^{-1}) \times \\
&\qquad \times \left(f_\alpha\left(\frac{z}{z+1}\right) - \left(\frac{z}{z+1}\right)^{-2\nu-1} \eta(S) f_\alpha\left(\frac{-1}{\frac{z}{z+1}}\right)\right).
\end{aligned}$$

Since $\frac{z}{z+1} = 1 - \frac{1}{z+1}$ and $f_\alpha\left(1 - \frac{1}{z+1}\right) = \eta(T) f_\alpha\left(\frac{-1}{z+1}\right)$ we see that the two middle summands cancel out. It remains

$$\begin{aligned}
&\eta(T) f_\alpha(z) - z^{-2\nu-1}\eta(ST^{-1}S) f_\alpha\left(\frac{-z-1}{z}\right) = \\
&= \ \eta(T) f_\alpha(z) - z^{-2\nu-1}\eta(ST^{-1}ST^{-1}) f_\alpha\left(\frac{-1}{z}\right) \\
&= \ \eta(T)\left(f_\alpha(z) - z^{-2\nu-1}\eta(S) f_\alpha\left(\frac{-1}{z}\right)\right) \\
&= \ \eta(T)\psi_\alpha(z).
\end{aligned}$$

Here we have used $ST^{-1}ST^{-1} = TS$. This proves that ψ_α satisfies the functional equation (8).

Next, if $\nu \in \frac{1}{2} + \mathbb{Z}$, then one sees that $\psi_\alpha(z) + z^{-2\nu-1}\eta(S)\psi_\alpha\left(\frac{-1}{z}\right)$ equals zero and so ψ_α lies in $\Psi_{\nu,\eta}$. If $\nu \notin \frac{1}{2} + \mathbb{Z}$ then recall that we take the standard

branch of the logarithm to define $z^{-2\nu-1}$. For $\psi(-1/z)$ one then takes a complimentary branch and one gets the inversion formula

$$f_\alpha(z) \;=\; \frac{1}{1+e^{\pm 2\pi i\nu}}\left(\psi_\alpha(z) + z^{-2\nu-1}\eta(S)\psi_\alpha\left(\frac{-1}{z}\right)\right) \tag{25}$$

for $z \in \mathbb{H}^\pm$. This proves $B\alpha \in \Psi_{\nu,\eta}$ and it only remains to show that the Bruggeman transform is surjective. But a simple calculation, similar to the one given above shows that for a holomorphic function $\psi\colon \mathbb{C} \setminus (-\infty,0] \to V_\eta$ satisfying (8) the function $f\colon \mathbb{C}\setminus\mathbb{R} \to V_\eta$, defined from ψ via the inversion formula (25), satisfies (4). If ψ satisfies (9), then f satisfies (5) and (6). In view of Proposition 2.2 this, finally, proves the claim.

□

3 Maaß wave forms

Recall the space $\mathcal{S}_{\nu,\eta}$ of *Maaß cusp forms* from (18) and consider a u in $\mathcal{S}_{\nu,\eta}$. Because of $u(z+N) = u(z)$ the function u has a Fourier series

$$u(z) \;=\; u(x+iy) \;=\; \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} A_k(y)\, e^{2\pi i \frac{k}{N}x}\, v_k$$

for some $v_k \in V_\eta$. The differential equation $\Delta u = (\frac{1}{4} - \nu^2)u$ implies a differential equation for $A_k(y)$ which implies that it must be a linear combination of I and K-Bessel functions. The fact that u is square integrable rules out the I-Bessel functions, so

$$A_k(y) \;=\; \sqrt{y} K_\nu\left(2\pi\frac{|k|}{N}y\right)$$

times a constant which we can assume to be 1 by multiplying it to v_k. By Theorem 3.2 of [4] it follows that the norms $\|v_k\|$ are bounded as $|k| \to \infty$. The functional equation $u(z+1) = \eta(T)u(z)$ is reflected in the fact that the v_k are eigenvectors of $\eta(T)$, since we get $\eta(T)v_k = e^{2\pi i\frac{k}{N}}v_k$. Now set

$$f_u(z) \;:=\; \begin{cases} \displaystyle\sum_{k>0} k^\nu e^{2\pi i\frac{k}{N}z} v_k, & \mathrm{Im}(z) > 0,\\ \displaystyle -\sum_{k<0} |k|^\nu e^{2\pi i\frac{k}{N}z} v_k, & \mathrm{Im}(z) < 0. \end{cases} \tag{26}$$

From the construction it is clear that f_u satisfies (4) - (6), i.e. $f_u \in \mathcal{F}_\eta$. It will play the role of our earlier f_α (cf. Proposition 2.2), so we define

$$\psi_u(z) \ := \ f_u(z) - z^{-2\nu-1}\eta(S) f_u\left(\frac{-1}{z}\right). \tag{27}$$

Lemma 3.1 *For* $\mathrm{Re}\nu > -\frac{1}{2}$ *the equation (26) and (27) define linear maps*

$$\begin{array}{rcl} \mathcal{S}_{\nu,\eta} & \to & \mathcal{F}_{\nu,\eta} \\ u & \mapsto & f_u \end{array} \quad \textit{and} \quad \begin{array}{rcl} \mathcal{S}_{\nu,\eta} & \to & \Psi^o_{\nu,\eta}. \\ u & \mapsto & \psi_u \end{array}$$

Proof: To prove that $f_u \in \mathcal{F}_{\nu,\eta}$ we will need the following two Dirichlet series. For $\varepsilon = 0, 1$ set

$$L_\varepsilon(u,s) \ := \ \sum_{k\neq 0} \mathrm{sign}(k)^\varepsilon \left(\frac{N}{|k|}\right)^s v_k.$$

We will relate L_0 and L_1 to u by the Mellin transform. For this let

$$u_0(y) \ = \ \frac{1}{\sqrt{y}} u(iy), \qquad u_1(y) \ = \ \frac{\sqrt{y}}{2\pi i} u_x(iy), \tag{28}$$

where $u_x = \frac{\partial}{\partial x} u$. Next define

$$\hat{L}_\varepsilon(u,s) \ := \ \int_0^\infty u_\varepsilon(y) y^s \frac{dy}{y}. \tag{29}$$

Plugging in the Fourier series of u and using the fact that

$$\int_0^\infty K_\nu(2\pi y) y^s \frac{dy}{y} \ = \ \Gamma_\nu(s) \ := \ \frac{1}{4\pi^s}\Gamma\left(\frac{s-\nu}{2}\right)\Gamma\left(\frac{s+\nu}{2}\right),$$

we get

$$\hat{L}_0(u,s) \ = \ \Gamma_\nu(s) L_0(u,s),$$

and similarly,

$$\hat{L}_1(u,s) \ = \ \Gamma_\nu(s+1) L_1(u,s).$$

On the other hand, the usual process of splitting the Mellin integral and using the functional equations

$$u_\varepsilon\left(\frac{1}{y}\right) \ = \ (-1)^\varepsilon y \, \eta(S) \, u_\varepsilon(y), \qquad \varepsilon = 0, 1,$$

(which can be checked using the Taylor series of u), one gets that $\hat{L}_\varepsilon$ extends to an entire function and satisfies the functional equation,

$$\hat{L}_\varepsilon(u,s) \;=\; (-1)^\varepsilon\,\eta(S)\,\hat{L}_\varepsilon(u,1-s).$$

With a similar, even easier computation one gets

$$\int_0^\infty y^s\left(f_u(iy)-(-1)^\varepsilon f_u(-iy)\right)\frac{dy}{y} \;=\; \frac{\Gamma(s)N^\nu}{(2\pi)^s}L_\varepsilon(u,s-\nu).$$

This implies that the Mellin transforms $M^\pm f(s) := \int_0^\infty y^s f(\pm iy)\,\frac{dy}{y}$ can be calculated as

$$\begin{aligned} M^\pm f(s) &= \pm\tfrac{\Gamma(s)N^\nu}{2(2\pi)^s}\left(L_0(u,s-\nu)\pm L_1(u,s-\nu)\right)\\ &= \pm N^\nu\pi^{-\nu-\frac{3}{2}}\Gamma\left(\tfrac{s+1}{2}\right)\Gamma\left(\tfrac{2\nu+2-s}{2}\right)\sin\pi\left(\nu+1-\tfrac{s}{2}\right)\,\hat{L}_0(u,s-\nu)\\ &\quad +N^\nu\pi^{-\nu-\frac{1}{2}}\Gamma\left(\tfrac{s}{2}\right)\Gamma\left(\tfrac{2\nu+1-s}{2}\right)\sin\pi\left(\nu+\tfrac{1}{2}-\tfrac{s}{2}\right)\,\hat{L}_1(u,s-\nu). \end{aligned}$$

The last identity follows from the standard equations

$$\Gamma\left(\tfrac{x}{2}\right)\Gamma\left(\tfrac{x+1}{2}\right) \;=\; \Gamma(x)2^{1-x}\sqrt{\pi}, \qquad \Gamma(x)\Gamma(1-x) \;=\; \tfrac{\pi}{\sin\pi x}.$$

Thus the Mellin transform $M^\pm f(s)$ is seen to be holomorphic for $\mathrm{Re}(s)>0$ and rapidly decreasing on any vertical strip. The Mellin inversion formula yields for $C>0$,

$$f_u(\pm iy) \;=\; \frac{1}{2\pi i}\int_{\mathrm{Re}\,(s)=C} y^{-s}M^\pm f_u(s)\,ds.$$

This extends to any $z\in\mathbb{C}\smallsetminus\mathbb{R}$ to give

$$f_u(z) \;=\; \frac{1}{2\pi i}\int_{\mathrm{Re}\,(s)=C} e^{\pm\frac{\pi}{2}is}z^{-s}\,M^\pm f_u(s)\,ds$$

for $z\in\mathbb{H}^\pm$. For $0<C<2\mathrm{Re}\nu+1$ (here we need $\mathrm{Re}\nu>-\frac{1}{2}$) it follows that

$$\psi_u(z) \;=\frac{1}{2\pi i}\int_{\mathrm{Re}\,(s)=C}\left(e^{\pm\frac{\pi}{2}is}z^{-s}-e^{\mp\frac{\pi}{2}is}z^{-2\nu-1}z^s\eta(S)\right)M^\pm f_u(s)\,ds.$$

Writing this as the difference of two integrals, substituting s in the second integral with $2\nu+1-s$ and shifting the contour we arrive at the formula

$$\begin{aligned} \tfrac{1}{2\pi i}\textstyle\int_{\mathrm{Re}\,(s)=C} \quad &\Big(e^{\pm\frac{\pi}{2}is}z^{-s}M^\pm f_u(s)\\ &-e^{\mp\frac{\pi}{2}i(2\nu+1-s)}z^{-s}\eta(S)M^\pm f_u(2\nu+1-s)\Big)\,ds. \end{aligned} \tag{30}$$

for ψ_u. Using the identities

$$\begin{aligned} \pm e^{\pm\frac{\pi}{2}is}\cos\pi\left(\nu+\frac{1}{2}-\frac{s}{2}\right)\mp e^{\mp\frac{\pi}{2}i(2\nu+1-s)}\cos\pi\frac{s}{2} &= i\sin\pi\left(\nu+\frac{1}{2}\right), \\ e^{\pm\frac{\pi}{2}is}\sin\pi\left(\nu+\frac{1}{2}-\frac{s}{2}\right)+e^{\mp\frac{\pi}{2}i(2\nu+1-s)}\sin\pi\frac{s}{2} &= \sin\pi\left(\nu+\frac{1}{2}\right), \end{aligned}$$

and the functional equation of $\hat{L}_\varepsilon$ we see that the integrand of (30) equals

$$\begin{aligned} z^{-s}N^\nu\sin\pi\left(\nu+\tfrac{1}{2}\right)\quad &\Big[\pi^{-\nu-\frac{3}{2}}\Gamma\left(\tfrac{s+1}{2}\right)\Gamma\left(\tfrac{2\nu+2-s}{2}\right)i\,\hat{L}_0(u,s-\nu)+ \\ &\quad+\pi^{-\nu-\frac{1}{2}}\Gamma\left(\tfrac{s}{2}\right)\Gamma\left(\tfrac{2\nu+1-s}{2}\right)\,\hat{L}_1(u,s-\nu)\Big]. \end{aligned} \tag{31}$$

Since this expression is independent of whether z lies in $\mathbb{H}^+$ or $\mathbb{H}^-$, it follows that $f_u(z)-z^{-2\nu-1}\eta(S)f_u\left(\frac{-1}{z}\right)$ extends to a holomorphic function on $\mathbb{C}\smallsetminus(-\infty,0]$, i.e., the function f_u indeed lies in the space $\mathcal{F}_{\nu,\eta}$. The linearity of the map is clear.

It remains to show that $\psi_u\in\Psi^o_{\nu,\eta}$. Note that in view of $f_u\in\mathcal{F}_{\nu,\eta}$ Proposition 2.2 shows that the function $z\mapsto(1+z^2)^{\nu+\frac{1}{2}}f_u(z)$ represents a hyperfunction $\alpha_u\in A^{-\omega}_{\nu,\eta}$. Then, according to Proposition 2.3 we have $\psi_u=B(\alpha_u)$ so ψ_u satisfies (9). The asymptotic property (19) now follows from the integral representation (30) with the C chosen there. More precisely, the bound $O(|z|^{-C})$ follows directly from (31) since the integrant divided by z^{-s} is of π-exponential decay, whereas the $O(1)$-bound is obtained by moving the contour slightly to the left of the imaginary axis picking up the residue at 0 which is proportional to 1 (see [7], §I.4 for more details on this type of argument). $\square$

Lemma 3.2 *For $0\neq k\in\mathbb{Z}$ let α_k be the hyperfunction on $\mathbb{P}_1(\mathbb{R})$ represented by $(1+z^2)^{\nu+\frac{1}{2}}f_k(z)$ with*

$$f_k(z)=\begin{cases}\operatorname{sign}(k)\cdot e^{2\pi i\frac{k}{N}z} & \textit{for}\ \operatorname{sign}(k)\cdot\operatorname{Im}(z)>0\\ 0 & \textit{for}\ \operatorname{sign}(k)\cdot\operatorname{Im}(z)<0.\end{cases}$$

Then we have that

$$\left\langle \pi_{-\nu}\begin{pmatrix}\sqrt{b} & \frac{a}{\sqrt{b}}\\ 0 & \frac{1}{\sqrt{b}}\end{pmatrix}\varphi_0,\alpha_k\right\rangle$$

equals

$$2\,\mathrm{sign}(k)\left(\frac{N}{|k|}\right)^{\nu}\frac{\pi^{-\nu-\frac{1}{2}}}{\Gamma(\frac{1}{2}-\nu)}\sqrt{b}\;K_\nu\left(2\pi\frac{|k|}{N}b\right)e^{2\pi i\frac{k}{N}a},$$

where is K_ν the K-Bessel function with parameter ν.

Proof: For this we will need the following identity (cf. [1], §4 or [9], p.136)

$$\int_{-\infty}^{\infty}\left(\frac{1}{y^2+(\tau-x)^2}\right)^{\frac{1}{2}-\nu}e^{2\pi ik\tau}\;dt=\frac{2\pi^{\frac{1}{2}-\nu}|k|^{-\nu}}{\Gamma\left(\frac{1}{2}-\nu\right)}y^\nu K_\nu(2\pi|k|y)e^{2\pi ikx}. \quad (32)$$

Note that $g=\begin{pmatrix}\sqrt{b} & \frac{a}{\sqrt{b}}\\ 0 & \frac{1}{\sqrt{b}}\end{pmatrix}$ satisfies $g\cdot i=a+ib$. Therefore, by abuse of notation, we write $P(\alpha_k)(a+ib)$ for $\left\langle \pi_{-\nu}\begin{pmatrix}\sqrt{b} & \frac{a}{\sqrt{b}}\\ 0 & \frac{1}{\sqrt{b}}\end{pmatrix}\varphi_0,\alpha_k\right\rangle$. According to [1], §4, we can calculate

$$\begin{aligned}
P(\alpha_k)(a+ib) &= \left\langle\left(\frac{1+x^2}{b+\left(\frac{x}{\sqrt{b}}-\frac{a}{\sqrt{b}}\right)^2}\right)^{-\nu+\frac{1}{2}},\alpha_k\right\rangle\\
&= b^{-\nu+\frac{1}{2}}\left\langle\left(\frac{1+x^2}{b^2+(x-a)^2}\right)^{-\nu+\frac{1}{2}},(1+z^2)^{\nu+\frac{1}{2}}f_k(z)\right\rangle\\
&= \mathrm{sign}(k)b^{-\nu+\frac{1}{2}}\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{b^2+(x-a)^2}\right)^{-\nu+\frac{1}{2}}e^{2\pi i\frac{k}{N}x}\,dx\\
&= 2\,\mathrm{sign}(k)\left(\frac{N}{|k|}\right)^{\nu}\frac{\pi^{-\nu-\frac{1}{2}}}{\Gamma(\frac{1}{2}-\nu)}\sqrt{b}\;K_\nu\left(2\pi\frac{|k|}{N}b\right)e^{2\pi i\frac{k}{N}a},
\end{aligned}$$

where in the last step we have used (32). □

Theorem 3.3 (Lewis transform; cf. [7], Thm. 1.1) *For $\nu\notin\frac{1}{2}+\mathbb{Z}$ and $\mathrm{Re}\nu>-\frac{1}{2}$ the Lewis transform is a bijective linear map from the space of Maaß cusp forms $\mathcal{S}_{\nu,\eta}$ to the space $\Psi^o_{\nu,\eta}$ of period functions.*

Proof: We begin by showing that the Lewis transform is injective on $\mathcal{S}_{\nu,\eta}$. This will be done by proving that we can recover u from ψ_u, where we use

the notation from Lemma 3.1. The hypothesis $\nu \notin \frac{1}{2}+\mathbb{Z}$ guarantees that we can recover f_u from ψ_u via a simple algebraic manipulation (cf. the proof of Proposition 2.3). Thus it suffices to express u in terms of α_u. But applying Lemma 3.2 to the summands in the defining formula (26) for f_u, we obtain

$$\begin{aligned} P\alpha_u(a+ib) &= \tfrac{2\,N^\nu}{\pi^{\nu+\frac{1}{2}}\Gamma(\frac{1}{2}-\nu)}\sqrt{b}\,\textstyle\sum_{k\neq 0} K_\nu\left(2\pi\frac{|k|}{N}b\right)e^{2\pi i\frac{k}{N}a}v_k \\ &= \tfrac{2\,N^\nu}{\pi^{\nu+\frac{1}{2}}\Gamma(\frac{1}{2}-\nu)}u(a+ib). \end{aligned} \tag{33}$$

It remains to show that $L(\mathcal{S}_{\nu,\eta})=\Psi^o_{\nu,\eta}$. To do this pick $\psi\in\Psi^o_{\nu,\eta}$. According to Propositions 2.2 and 2.3 we can find a hyperfunction $\alpha\in A^\Gamma_{\nu,\eta}$ represented by the function $(1+z^2)^{\nu+\frac{1}{2}}f$ with $f\in\mathcal{F}_{\nu,\eta}$ such that

$$\begin{aligned} \psi(z) &= f(z)-z^{-2\nu-1}\eta(S)f\left(-\frac{1}{z}\right), \\ f(z) &= \frac{1}{1+e^{\pm 2\pi i\nu}}\left(\psi(z)+z^{-2\nu-1}\eta(S)\psi\left(\frac{-1}{z}\right)\right) \end{aligned}$$

for $z\in\mathbb{H}^\pm$. The function f admits a Fourier expansion of the form

$$f(z) = \begin{cases} \dfrac{1}{2}v_0+\displaystyle\sum_{k=1}^{\infty}e^{2\pi i\frac{k}{N}z}v_k, & z\in\mathbb{H}^+, \\ -\dfrac{1}{2}v_0-\displaystyle\sum_{k=1}^{\infty}e^{-2\pi i\frac{k}{N}z}v_{-k}, & z\in\mathbb{H}^-. \end{cases}$$

The asymptotic property (19) of ψ implies that

$$\begin{aligned} \psi(z) &= O(|z|^{-C}) \\ z^{-2\nu-1}\eta(S)\psi\left(-\tfrac{1}{z}\right) &= O(|z|^{-2\nu-1}) \end{aligned}$$

for $\zeta\in\mathbb{C}\setminus(-\infty,0]$. Since $2\nu+1>0$ this implies that there is a constant $\epsilon>0$ such that

$$f(x+iy)=O(|y|^{-\varepsilon})$$

locally uniformly in x. Since f is periodic, this shows $v_0=0$. Note that $K_\nu(t)\sim e^{-t}\sqrt{\frac{\pi}{2t}}$. Therefore we have

$$A_k(y) = \sqrt{y}K_\nu\left(2\pi\frac{|k|}{N}y\right)\sim e^{-2\pi\frac{|k|}{N}y}\sqrt{\frac{N}{4\,|k|}} \tag{34}$$

uniformly in k, which implies that

$$u(z) \;:=\; u(x+iy) \;:=\; \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} A_k(y)\, e^{2\pi i \frac{k}{N} x}\, v_k$$

defines a smooth function on $\mathbb{H}^+$. Taking the derivatives termwise, we see that u satisfies (17), i.e. is contained in the range of the Poisson transform. Now Lemma 3.2 shows that

$$\left\langle \pi_{-\nu} \begin{pmatrix} \sqrt{b} & \frac{a}{\sqrt{b}} \\ 0 & \frac{1}{\sqrt{b}} \end{pmatrix} \varphi_0, \alpha \right\rangle = u$$

and (20) implies $u \in \tilde{\mathcal{M}}_{\nu,\eta}$. Note that (18) is a consequence of $v_0 = 0$. Thus in order to show that $\psi \in L(\mathcal{S}_{\nu,\eta})$, it only remains to show that u satisfies (16). But (34) implies that u rapidly decreases towards the cusp and hence the finite volume of the fundamental domain proves the square integrability of u. $\square$

As a consequence of this proof we see that for η the trivial representation, our Lewis transform coincides with $\frac{1}{2}\pi^{\nu+\frac{1}{2}}\Gamma\left(\frac{1}{2}-\nu\right)$ times the one given in [7].

4 Characterizing period functions on $\mathbb{R}^+$

Let $T' = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = (STS^{-1})^{-1}$.

Lemma 4.1 (cf. [7], §III.3) *If a smooth function $\psi\colon (0,\infty) \to V_\eta$ satisfies (22) with $\nu \notin \frac{1}{2}+\mathbb{Z}$, then it has the following asymptotic expansions:*

$$\begin{aligned} \psi(x) &\underset{x\to 0}{\sim} x^{-2\nu-1} Q_0\big(\tfrac{1}{x}\big) + \sum_{m=-1}^{\infty} C_m^* x^m, \\ \psi(x) &\underset{x\to \infty}{\sim} x^{-2\nu-1} Q_\infty\big(\tfrac{1}{x}\big) + \sum_{m=-1}^{\infty} (-1)^m C_m^* x^{-m-2\nu-1}, \end{aligned}$$

where the $Q_0, Q_\infty\colon R \to \mathbb{C}$ are smooth functions with

$$\begin{aligned} Q_0(x+1) &= \eta(T')Q_0(x), \\ Q_\infty(x+1) &= \eta(T)Q_\infty(x), \end{aligned}$$

and the C^*_m *can be calculated from the Taylor coefficients* $C_m := \frac{1}{m!}\psi^{(m)}(1) \in V_\eta$ *of* ψ *in* 1 *via*

$$C^*_m = \frac{1}{m+2\nu+1}\sum_{k=0}^{M}(-1)^m B_k \binom{m+2\nu+1}{k} C_{m-1-k}. \tag{35}$$

Here the B_k *are the Bernoulli numbers. If* ψ *is real analytic, then so are* Q_0 *and* Q_∞*.*

Proof: For $\mathrm{Re}\nu > 0$ set

$$Q_0(x) \;:=\; x^{-2\nu-1}\psi\left(\frac{1}{x}\right) - \sum_{n=0}^{\infty}(n+x)^{-2\nu-1}\eta\big(T(T')^n\big)^{-1}\psi\left(1+\frac{1}{n+x}\right)$$

and

$$Q_\infty(x) \;:=\; \psi(x) - \sum_{n=1}^{\infty}(n+x)^{-2\nu-1}\eta\big(T'T^{n-1}\big)^{-1}\psi\left(1-\frac{1}{n+x}\right).$$

Then we have

$$
\begin{aligned}
&Q_0(x+1) - \eta(T')Q_0(x) = \\
= \quad & (x+1)^{-2\nu-1}\psi\left(\frac{1}{x+1}\right) - x^{-2\nu-1}\eta(T')\psi\left(\frac{1}{x}\right) \\
& -\sum_{n=0}^{\infty}(n+1+x)^{-2\nu-1}\eta(T(T')^n)^{-1}\psi\left(1+\frac{1}{n+1+x}\right) \\
& +\sum_{n=0}^{\infty}(n+x)^{-2\nu-1}\eta(T')\eta(T(T')^n)^{-1}\psi\left(1+\frac{1}{n+x}\right) \\
= \quad & (x+1)^{-2\nu-1}\psi\left(\frac{1}{x+1}\right) - x^{-2\nu-1}\eta(T')\psi\left(\frac{1}{x}\right) \\
& -\sum_{n=1}^{\infty}(n+x)^{-2\nu-1}\eta(T(T')^{n-1})^{-1}\psi\left(1+\frac{1}{n+x}\right) \\
& +\sum_{n=0}^{\infty}(n+x)^{-2\nu-1}\eta(T(T')^{n-1})^{-1}\psi\left(1+\frac{1}{n+x}\right) \\
= \quad & (x+1)^{-2\nu-1}\psi\left(\frac{1}{x+1}\right) - x^{-2\nu-1}\eta(T')\psi\left(\frac{1}{x}\right) \\
& +x^{-2\nu-1}\eta(T(T')^{-1})^{-1}\left(\eta(T)\psi\left(\frac{1}{x}\right)\right. \\
& \left. -\left(1+\frac{1}{x}\right)^{-2\nu-1}\eta(ST^{-1})\psi\left(\frac{\frac{1}{x}}{1+\frac{1}{x}}\right)\right) \\
= \quad & 0,
\end{aligned}
$$

since $T^{-1}ST^{-1} = (T')^{-1}$. Similarly we calculate

$$
\begin{aligned}
&Q_\infty(x+1) - \eta(T)Q_\infty(x) = \\
&= \psi(x+1) - \eta(T)\psi(x) \\
&\quad - \sum_{n=1}^{\infty} (n+1+x)^{-2\nu-1} \eta(T'T^{n-1})^{-1} \psi\left(1 - \frac{1}{n+1+x}\right) \\
&\quad + \sum_{n=1}^{\infty} (n+x)^{-2\nu-1} \eta(T)\eta(T'T^{n-1})^{-1} \psi\left(1 - \frac{1}{n+x}\right) \\
&= \psi(x+1) - \eta(T)\psi(x) \\
&\quad - \sum_{n=2}^{\infty} (n+x)^{-2\nu-1} \eta(T'T^{n-2})^{-1} \psi\left(1 - \frac{1}{n+x}\right) \\
&\quad + \sum_{n=1}^{\infty} (n+x)^{-2\nu-1} \eta(T'T^{n-2})^{-1} \psi\left(1 - \frac{1}{n+x}\right) \\
&= \psi(x+1) - \eta(T)\psi(x) \\
&\quad + (1+x)^{-2\nu-1} \eta(T'T^{-1})^{-1} \psi\left(1 - \frac{1}{x+1}\right) \\
&= 0.
\end{aligned}
$$

For general ν we write

$$
\begin{aligned}
Q_0(x) &:= x^{-2\nu-1}\psi\left(\frac{1}{x}\right) - \sum_{m=0}^{M} C_m \zeta(m+2\nu+1, x) \\
&\quad - \sum_{n=0}^{\infty} (n+x)^{-2\nu-1} \eta\big(T(T')^n\big)^{-1} \left(\psi\left(1 + \frac{1}{n+x}\right) - \sum_{m=0}^{M} \frac{C_m}{(n+x)^m}\right) \\
Q_\infty(x) &:= \psi(x) - \sum_{m=0}^{M} (-1)^m C_m \zeta(m+2\nu+1, x+1) \\
&\quad - \sum_{n=1}^{\infty} (n+x)^{-2\nu-1} \eta\big(T'T^n\big)^{-1} \left(\psi\left(1 - \frac{1}{n+x}\right) - \sum_{m=0}^{M} \frac{C_m}{(n+x)^m}\right)
\end{aligned}
$$

with the Hurwitz zeta function $\zeta(a,x) := \sum_{n=0}^{\infty} \frac{1}{(n+x)^a}$. Since the Hurwitz zeta function satisfies

$$
\zeta(a,x) \underset{x\to\infty}{\sim} \frac{1}{a-1} \sum_{k\geq 0} (-1)^k B_k \binom{k+a-2}{k} x^{-a-k+1} \tag{36}
$$

we find

$$\begin{aligned}\psi(x) &= x^{-2\nu-1}Q_0(x^{-1}) + \sum_{m=0}^{M} C_m \zeta(m+2\nu+1, x^{-1})x^{-2\nu-1} \\ &\quad + \underbrace{\sum_{n=0}^{\infty}(x^{-1}+n)^{-2\nu-1}\left(\psi\left(1+\frac{1}{n+x^{-1}}\right) - \sum_{m=0}^{M}\frac{C_m}{(n+x^{-1})^n}\right)}_{=O(x^{2\nu+1+M})}.\end{aligned}$$

From this one derives the asymptotics for $x \to 0$ using (36), see [7], §III.3 for details. The asymptotics for $x \to \infty$ is shown analogously and the last claim is obvious. □

Remark 4.2 (i) If $\psi(x) = o(x^{\min(1,2\mathrm{Re}\,\nu+1)})$ for $x \to 0$, then $Q_0 = 0$ by periodicity, i.e., ψ is an eigenfunction for the *transfer operator*

$$\mathcal{L}_0\psi(x) := x^{-2\nu-1}\sum_{n=0}^{\infty}(n+x^{-1})^{-2\nu-1}\eta\big(T(T')^n\big)^{-1}\psi\left(1+\tfrac{1}{n+x^{-1}}\right).$$

Moreover we have $C^*_{-1} = 0$.

(ii) If $\psi(x) = o(x^{\min(0,\mathrm{Re}\,\nu)})$ for $x \to \infty$, then $Q_\infty = 0$, i.e., ψ is an eigenfunction for the *transfer operator*

$$\mathcal{L}_\infty\psi(x) := \sum_{n=1}^{\infty}(n+x)^{-2\nu-1}\eta\big(T'T^n\big)^{-1}\psi\left(1+\tfrac{1}{n+x}\right).$$

Moreover we have $C^*_{-1} = 0$.

(iii) If $C^*_{-1} = 0$, then $C_0 = 0$, and if $Q_0 = Q_\infty = 0$ we have the equations

$$\psi(x) = x^{-2\nu-1}\sum_{n=0}^{\infty}(n+x^{-1})^{-2\nu-1}\eta\big(T(T')^n\big)^{-1}\psi\left(1+\tfrac{1}{n+x^{-1}}\right) \tag{37}$$

$$\psi(x) = \sum_{n=1}^{\infty}(n+x)^{-2\nu-1}\eta\big(T'T^n\big)^{-1}\psi\left(1+\tfrac{1}{n+x}\right). \tag{38}$$

In this case, we can analytically extend ψ to $\mathbb{C} \setminus (-\infty, 0]$ via

$$\psi(z) := \sum_{\gamma\in Q_n}(\psi|_{\nu,\eta}\gamma)(z),$$

where Q is the semigroup generated by T and T', Q_n is the set of T–T'-words of length n in Q, and

$$(\psi|_{\nu,\eta}\gamma)(z) := (cz+d)^{-2\nu-1}\eta(\gamma)^{-1}\psi(\gamma\cdot z) \tag{39}$$

is a well defined right semigroup action (cf. [3], § 3, and [7], §III.3). The analytically continued function ψ still satisfies (37) and (38). Therefore we can mimick the proof of Lemma 4.1 and use the Taylor expansion in 1 to find

$$\psi(z) = \sum_{m=1}^{M} C_m \zeta(m+2\nu+1, z^{-1}) z^{-2\nu-1} + O\big(|\zeta(2\nu+M+2, z^{-1})|\big) \tag{40}$$

for $|z| \to 0$ and

$$\psi(z) = \sum_{m=1}^{M} (-1)^m C_m \zeta(m+2\nu+1, z+1) + O\big(|\zeta(2\nu+M+2, z)|\big), \tag{41}$$

for $|z| \to \infty$. Now we use the following version of (36) which can be found in [2], § 1.18:

$$\begin{aligned} \zeta(a,z) \quad &= \quad z^{1-a}\tfrac{\Gamma(a-1)}{\Gamma(a)} + \tfrac{1}{2}z^{-a} + \textstyle\sum_{n=1}^{N} B_{2n}\tfrac{\Gamma(a+2n-1)}{\Gamma(a)(2n)!} z^{1-2n-a} \\ &\quad\ \ +O(|z|^{-2N-1-a}) \end{aligned} \tag{42}$$

for $\mathrm{Re}\, a > 1$ and $z \in \mathbb{C} \setminus (-\infty, 0]$. Then (40) and (41) result in

$$\psi(z) = O(1) \quad \text{for } |z| \to 0 \tag{43}$$

and

$$\psi(z) = O(|z|^{-2\nu-1}) \quad \text{for } |z| \to \infty. \tag{44}$$

$\square$

Remark 4.3 One can use the slash action (39) to rewrite the real version (22) of the Lewis equation in the form

$$\psi = \psi|_{\nu,\eta}T + \psi|_{\nu,\eta}T'.$$

$\square$

Theorem 4.4 (cf. [7], Thm. 2) *Suppose that* $\mathrm{Re}\nu > -\frac{1}{2}$. *Then*

$$\Psi^{\mathbb{R}}_{\nu,\eta} = \{\psi|_{(0,\infty)} : \psi \in \Psi^{o}_{\nu,\eta}\}.$$

Proof: Note first that property (19) of $\psi \in \Psi^{o}_{\nu,\eta}$ trivially implies (23) and (24) for $\psi|_{(0,\infty)}$. Therefore it only remains to show that each element of $\Psi^{\mathbb{R}}_{\nu,\eta}$ occurs as the restriction of some $\psi \in \Psi^{o}_{\nu,\eta}$. To this end we fix a $\tilde{\psi} \in \Psi^{\mathbb{R}}_{\nu,\eta}$. Since (23) and (24) hold for $\tilde{\psi}$, we can apply Remark 4.2 to it. Thus $\tilde{\psi}$ has an analytic continuation to $\mathbb{C} \setminus (-\infty, 0]$ (still denoted by $\tilde{\psi}$) and the asymptotics (43) and (44) shows that $\tilde{\psi}$ indeed satisfies (19). $\square$

5 A Converse Theorem

Theorem 5.1 *Let* $v_k \in V_\eta$ *for* $k \in \mathbb{Z} \setminus \{0\}$ *such that* $\eta(T)v_k = e^{2\pi i \frac{k}{N}} v_k$ *and that the two Dirichlet series*

$$L_\varepsilon(s) \; = \; \sum_{k\neq 0} \mathrm{sgn}(k)^\varepsilon \left(\frac{N}{|k|}\right)^s v_k, \qquad \varepsilon = 0, 1$$

converge for $\mathrm{Re}(s) >> 0$. *Assume that* $\hat{L}_\varepsilon(s) = \Gamma_\nu(s+\varepsilon) L_\varepsilon(s)$ *extends to an entire function with*

$$\hat{L}_\varepsilon(s) \; = \; (-1)^\varepsilon \, \eta(S) \, \hat{L}_\varepsilon(1-s).$$

Then the function u *given by*

$$u(z) \; = \; \sum_{k\neq 0} \sqrt{y} K_\nu \left(2\pi \frac{|k|}{N} y\right) e^{2\pi i \frac{k}{N} x} \, v_k$$

lies in $\mathcal{S}_{\nu,\eta}$.

Proof: The Dirichlet series give rise to an inverse Mellin transform f as in Section 3. Now follow the argumentation of that section. $\square$

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